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Algebraic properties of a special class of birth–death master equations

L V Il'ichov

Institute for Automation and Electrometry SB RAS, Novosibirsk State University, Novosibirsk 630090, Russia

E-mail: shalagin@iae.nsk.su

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Abstract
We give a complete description of a simple special class of stochastic models (birth–death master equations). Each equation of this class can be easily solved by the method suggested by supersymmetric quantum mechanics. Using the quadratic Lyapunov functional, one can transform the master equation to a Schrödinger-type equation with a self-adjoint Hamiltonian in imaginary time. Then the hidden symmetries of the Hamiltonian are investigated. The whole set of models can be decomposed to four subsets with natural algebraic classification.

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1. Introduction

In the present work we deal with birth–death master equations (see, e.g. [1] chapter 7.1)

$$\dot{p}(n,t) = W_-(n+1)p(n+1,t) + W_+(n-1)p(n-1,t) - [W_+(n) + W_-(n)]p(n,t) \qquad (1.1)$$

where n ($n = 0, 1, 2, \ldots$) specifies the state of the considered system (number of particles, specimen, level number in a quantum system, etc) In every elementary act of a random process n changes by one. The probability $p(n, t)$ of finding the system at time t in the state n changes in accordance with equation (1.1), where $W_\pm(n)$ are the rates of the transitions $n \mapsto n \pm 1$.

Il'ichov [2] proposed a new method to find solutions to (1.1) resembling the application of supersymmetric quantum mechanics to the construction of the exactly solvable Schrödinger equations with so-called shape-invariant potentials [3][1]. Let us reproduce the main steps of

[1] There are a number of works where the methods of supersymmetric quantum mechanics are applied to kinetic problems: [4–9].

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the approach from [2]. First of all one should note that equation (1.1) possesses the Lyapunov functional

$$H(t) = \sum_n p_S^{-1}(n) p^2(n,t) \tag{1.2}$$

where $p_S(n)$ is the stationary probability distribution. Introducing the functional $H(t)$, we restrict our consideration to distributions $p(n, t)$ which make the rhs of (1.2) finite. In this case it is easy to show that

$$\dot{H}(t) \leqslant 0 \tag{1.3}$$

where equality is only realized for the stationary distribution. Note that considering the quantity $p_S^{-1/2}(n)p(n,t)$ as the nth component of a vector ψ from a real separable Hilbert space $\mathcal{H}'(0) = \mathrm{span}\{e_n; n = 0, 1, 2, \ldots\}$ (with the element e_n of orthonormal basis being associated with the nth state of the considered system), one can present $H(t)$ as the square norm of the vector. Being written for the vector $\psi(t)$, equation (1.1) reads

$$\begin{aligned}\dot{\psi}(n,t) = &-[\hat{K}\psi(t)](n)\\ &\equiv \sqrt{W_-(n)W_+(n-1)}\psi(n-1,t) + \sqrt{W_-(n+1)W_+(n)}\psi(n+1,t)\\ &-\left[W_+(n) + W_-(n)\right]\psi(n,t).\end{aligned} \tag{1.4}$$

In such a form, the kinetic operator $\hat{K}$, which is introduced in (1.4) is explicitly symmetric. It is convenient to assume $\hat{K} \in \mathcal{B}(\mathcal{H}'(0))$ (in the present work we deal mainly with finite dimensional spaces $\mathcal{H}'(0)$). The condition (1.3) may be written as

$$\dot{H}(t) = -2(\psi(t) \qquad \hat{K}\psi(t)) \leqslant 0. \tag{1.5}$$

This means that the operator $\hat{K}$ is positive semi-definite. It is known that under these conditions an operator $\hat{A} \in \mathcal{B}(\mathcal{H}'(0))$ can be found, such that

$$\hat{K} = \hat{A}^\dagger \hat{A}. \tag{1.6}$$

It follows from (1.5) that $\hat{A}$ annihilates the 'vacuum vector' $\psi_0 = \sum_n \sqrt{p_S(n)} e_n$ so that

$$\hat{A}\psi_0 = 0.$$

Of concern to us are the discrete spectrum $\{\lambda_i\}_{i=0,1,\ldots}$ $(0 = \lambda_0 < \lambda_1 < \cdots)$[2] and the set of corresponding eigenvectors $\{\psi_i\}_{i=0,1,\ldots}$ of $\hat{K}$:

$$\hat{K}\psi_i = \lambda_i \psi_i.$$

As is shown in [2] they can easily be obtained if one manages to find the set of objects

$$\{\lambda_i(k)\}_{i=0,1,\ldots}, \qquad \{\psi_i(k)\}_{i=0,1,\ldots}, \qquad \hat{A}(k), \qquad \mathcal{H}'(k)\ (k = 0, 1, 2, \ldots)$$

where $\psi_i(0) \equiv \psi_i$, $\lambda_i(0) \equiv \lambda_i \forall i$ $(0 = \lambda_0(k) < \lambda_1(k) < \cdots)$ $\forall k$, $\hat{A}(0) \equiv \hat{A}$ with the following properties. For any k the vector $\psi_i(k)$ belongs to the space $\mathcal{H}'(k)$ which is convenient to consider as a subspace of an including space $\mathcal{H}'$,

$$\mathcal{H}' = \sum_k \mathcal{H}'(k),$$

where $\mathcal{H}'(0) \supset \mathcal{H}'(1) \supset \ldots$. The operators $\hat{A}(k) \in \mathcal{B}(\mathcal{H}'(k), \mathcal{H}'(k+1))$ act as follows:

$$\hat{A}(k)\psi_i(k) = \sqrt{\lambda_i(k)}\psi_{i-1}(k+1) \qquad \hat{A}^\dagger(k)\psi_i(k+1) = \sqrt{\lambda_{i+1}(k)}\psi_{i+1}(k).$$

[2] The real nature of λ_i is evident. We also assume that the considered birth–death equations possess no cunning hidden symmetry which can make the discrete spectrum of $\hat{K}$ degenerate. One can find stronger reasons for this in the theory of orthogonal polynomials (see e.g. [10]) and the fact that the entries of the tridiagonal matrix $\hat{K}$ meet the conditions for Favard's theorem (pointed out by the referee to whom the author is indebted).

$\hat{A}(k)$ and $\hat{A}(k+1)$ obey the relation

$$\hat{A}(k)\hat{A}^\dagger(k) - \hat{A}^\dagger(k+1)\hat{A}(k+1) = \lambda_1(k)\hat{\mathbb{I}}_{\mathcal{H}'(k+1)} \tag{1.7}$$

i.e. $\lambda_i(k+1) = \lambda_{i+1}(k) - \lambda_1(k)$ for $i, k = 0, 1, \ldots$. This means that upon the lifting displacement the spectrum of $\hat{A}^\dagger(k+1)\hat{A}(k+1)$ coincides with that of $\hat{A}^\dagger(k)\hat{A}(k)$ except for the 'ground level'. In other words, for all k there should be $Ker\hat{A}^\dagger(k) = 0$, which is known as the condition of unbroken supersymmetry. Equation (1.7) allows one to find $\hat{A}(k+1)$ and $\lambda_1(k)$ starting from $\hat{A}(k)$. Solving the problem step by step, one is able to define the eigenvectors

$$\psi_i \equiv \psi_i(0) = \prod_{k=0}^{i-1} \frac{\hat{A}^\dagger(k)}{\sqrt{\lambda_{i-k}(k)}}\psi_0(i) \tag{1.8}$$

and the corresponding eigenvalues

$$\lambda_i \equiv \lambda_i(0) = \sum_{k=0}^{i-1} \lambda_1(k) \tag{1.9}$$

for $\hat{K}$. The normalized solution to the equation

$$\hat{A}(i)\psi_0(i) = 0$$

and the quantities $\lambda_{i-k}(k) = \lambda_i(0) - \lambda_k(0)$ occurs in (1.8).

In [2] the operators $\hat{A}(k)$ and $\hat{A}^\dagger(k)$ were used as

$$\hat{A}(k) = \hat{a}w_-(\hat{n}, k) - w_+(\hat{n}, k) \qquad \hat{A}^\dagger(k) = w_-(\hat{n}, k)\hat{a}^\dagger - w_+(\hat{n}, k) \tag{1.10}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the ordinary lowering and raising bosonic operators with respect to the basis $\{e_n\}_{n=0,1,\ldots,}$

$$(e_{n'}, \hat{a}e_n) = \sqrt{n}\delta_{n',n-1} \qquad (e_{n'}, \hat{a}^\dagger e_n) = \sqrt{n+1}\delta_{n',n+1}$$

and $w_\pm(\hat{n}, k)$ are real functions of $\hat{n} = \hat{a}^\dagger\hat{a}$ ($\hat{n}$: $e_n \mapsto ne_n$) and the step number k so that

$$[w_+(n,0)]^2 = W_+(n,0) \equiv W_+(n) \qquad n[w_-(n,0)]^2 = W_-(n,0) \equiv W_-(n).$$

A calculation of matrix elements shows that introducing the functions $W_+(n,k) \equiv [w_+(n,k)]^2$ and $W_-(n,k) \equiv n[w_-(n,k)]^2$ one can replace equation (1.7) with operators (1.10) by the following pair of conditions

$$\begin{aligned} W_-(n+1,k) + W_+(n,k) - W_-(n,k+1) - W_+(n,k+1) &= \lambda_1(k) \\ W_-(n,k)W_+(n,k) &= W_-(n,k+1)W_+(n-1,k+1). \end{aligned} \tag{1.11}$$

It is convenient to consider the quantities $W_\pm(n,k)$ as transition rates for an effective stochastic model of the kth step. Equations (1.11) are more convenient than the initial operator equation. In [2] a solution to (1.11) was found in the case of a model chemical reactor with cross-inversion of enantiomers. It will be shown below that this model is an element of a set of contiguous models realizing the most simple but nontrivial solutions to (1.11).

As a natural approach, equations (1.11) suggest a search of an ansatz which makes the second line in (1.11) trivially fulfilled. Then the first line provides explicit expressions for $W_\pm(n,k)$ as well as $\lambda_1(k)$. This will be done in the next section. It appears that the cases $\lambda_1(0) > \lambda_1(1)$, $\lambda_1(0) < \lambda_1(1)$ and $\lambda_1(0) = \lambda_1(1)$ should be considered separately, which is done in the sections 3, 4 and 5. An algebra of operators which acts in the space of eigenvectors may be associated with each case. Hence the considered solutions get a natural algebraic classification.

2. General relations

Let us try the following ansatz suggested by the model from [2]:

$$W_-(n,k) = W_-(n) \qquad W_+(n,k) = W_+(n+k). \tag{2.1}$$

Provided that this substitution is possible, the condition of unbroken supersymmetry is fulfilled for every step k for which the quantities $W_+(n+k)$ may be considered as transition rates for a consistent model. By (2.1), the first equation (1.11) gets the form

$$W_-(n+1) - W_+(n+1+k) = \lambda_1(k) + W_-(n) - W_+(n+k)$$

hence ($W_-(0) = 0$)

$$W_-(n) - W_+(n+k) = n\lambda_1(k) - W_+(k). \tag{2.2}$$

If we put $k = 0$ and $k = 1$ in the last equation, we get

$$W_+(n) = \frac{1}{2}n(n-1)\,(\lambda_1(0) - \lambda_1(1)) + nW_+(1) + (1-n)W_+(0). \tag{2.3}$$

By (2.2) taken for $k = 0$ and (2.3), we also have

$$W_-(n) = \frac{1}{2}n(n-1)\,(\lambda_1(0) - \lambda_1(1)) + nW_-(1)$$

where

$$W_-(1) = \lambda_1(0) + W_+(1) - W_+(0). \tag{2.4}$$

Then (2.2) implies:

$$\lambda_1(k) = \lambda_1(0) - k\,(\lambda_1(0) - \lambda_1(1))\,. \tag{2.5}$$

Thus the transition rates which realize the ansatz (2.1) are second-order polynomials of n. Each model of the considered set is specified by the four parameters—$\lambda_1(0)$, $\lambda_1(1) = \lambda_2(0) - \lambda_1(0)$, $W_+(0)$ and $W_+(1)$. However, these parameters should fulfil some relations which make the model consistent. We consider them in the following sections.

3. The case $\lambda_1(0) > \lambda_1(1)$

Note that the initial model with the transition rates $W_\pm(n)$ (the model of step zero) can in its turn be generated at an intermediate step by another model. Hence it is also worth considering the negative step numbers k. As will be shown below, this way is very convenient.

When $\lambda_1(0) > \lambda_1(1)$ (or, equivalently, $\lambda_1(0) > \lambda_2(0)/2$) then for sufficiently large k the rhs of (2.5) becomes negative. This is senseless. Hence in this situation k should be bounded from above. At the same time there is no lower bound for k. This restriction makes the finiteness of the set of states inevitable, i.e. there should be N such that

$$W_+(N-1) > 0 \qquad W_+(N) = 0. \tag{3.1}$$

In the opposite case, for any $k > 0$ in accordance with (2.1) one would get a consistent model with positive $\lambda_1(k)$, which is in conflict with the premise. One can also make sure that there should be $\min\{k\colon \lambda_1(k) \leqslant 0\} \geqslant N$. By (3.1), we have from (2.3)

$$\frac{1}{2}(\lambda_1(0) - \lambda_1(1)) = \frac{(N-1)W_+(0) - NW_+(1)}{N(N-1)} \tag{3.2}$$

and

$$W_+(n) = (N-n)\left[n\frac{W_+(1)}{N-1} + (1-n)\frac{W_+(0)}{N}\right]. \tag{3.3}$$

We assume implicitly that $N > 1$ (only in this case is it reasonable to consider the quantity $\lambda_1(1)$). The second null for $W_+(n)$ appears for

$$n_0 = \frac{(N-1)W_+(0)}{(N-1)W_+(0) - NW_+(1)} \geqslant N. \tag{3.4}$$

Note that this quantity does not have to be an integer. By (3.4), we get

$$(N-1)^2 W_+(0) \leqslant N^2 W_+(1)$$

and because of the positivity of the rhs of (3.2)

$$(N-1)W_+(0) > NW_+(1).$$

As a result we get the following condition:

$$\left(\frac{N-1}{N}\right)^2 W_+(0) \leqslant W_+(1) < \frac{N-1}{N} W_+(0). \tag{3.5}$$

Now we are going to get a condition for $\lambda_1(0)$. By the positivity of $W_-(1)$ in (2.4), we have

$$\lambda_1(0) > W_+(0) - W_+(1). \tag{3.6}$$

On the other hand, as has been pointed above, $\lambda_1(k)$ should be positive up to $k = N-1$, i.e.

$$\lambda_1(0) > (N-1)\left(\lambda_1(0) - \lambda_1(1)\right).$$

By this inequality and by (3.2), we have

$$\lambda_1(0) > 2\left(W_+(0) - W_+(1)\right) - \frac{2}{N} W_+(0). \tag{3.7}$$

If we assume that condition (3.7) is stronger than (3.6), then $W_+(1) < (N-2)W_+(0)/N$. Comparing this expression with the left inequality in (3.5), we come to the contradiction $N(N-2) > (N-1)^2$. Hence (3.6) is the defining condition on $\lambda_1(0)$. After the choice of $W_+(0)$, $W_+(1)$, and $\lambda_1(0)$ one can find $\lambda_1(1)$ from (3.2).

We are about to associate an algebra[3] with the chain of equations (1.7). The case $\lambda_1(0) > \lambda_1(1)$ is the simplest since the associated algebra appears to be a Lie algebra. The idea is to consider the step number k ($k = N, N-1, \ldots$) as a quantum number numerating the basis vector f_k of a Hilbert space $\mathcal{H}''$. Now we introduce the raising $\mathbf{b}^\dagger$ and lowering $\mathbf{b}$ operators in $\mathcal{H}''$,

$$\mathbf{b}^\dagger f_k = f_{k+1} \qquad \mathbf{b}^\dagger f_N = 0 \qquad \mathbf{b} f_k = f_{k-1}$$

and build the operators $\mathcal{J}_0$, $\mathcal{J}_\pm$, which act in $\mathcal{H}' \otimes \mathcal{H}''$ ($\mathcal{H}' = \sum_{k=-\infty}^{N} \mathcal{H}'(k)$),

$$\begin{aligned} &\mathcal{J}_0(\psi \otimes f_k) = (k+\alpha)(\psi \otimes f_k) \quad (\forall \psi \in \mathcal{H}') \\ &\mathcal{J}_+ = \beta \mathbf{b}^\dagger \hat{A}(\mathcal{J}_0 - \alpha) \qquad \mathcal{J}_- = \beta \hat{A}^\dagger(\mathcal{J}_0 - \alpha)\mathbf{b}, \end{aligned} \tag{3.8}$$

where α and β are some constants to be defined. Note that

$$[\mathcal{J}_\pm, \mathcal{J}_0] = \mp \mathcal{J}_\pm. \tag{3.9}$$

We have

$$\begin{aligned} \beta^{-2}[\mathcal{J}_+, \mathcal{J}_-] = {} & \hat{A}(\mathcal{J}_0 - \alpha - 1)\hat{A}^\dagger(\mathcal{J}_0 - \alpha - 1) - \hat{A}^\dagger(\mathcal{J}_0 - \alpha)\hat{A}(\mathcal{J}_0 - \alpha) \\ & + \hat{A}^\dagger(\mathcal{J}_0 - \alpha)\hat{A}(\mathcal{J}_0 - \alpha)\mathcal{P}_N \end{aligned} \tag{3.10}$$

[3] A similar method is used when considering the shape-invariant potentials in supersymmetric quantum mechanics (see e.g. the work [11]). Unfortunately this approach cannot be directly applied to our problems.

where $\mathcal{P}_k$ is the projector on f_k along $\mathcal{H}'$. When the rhs of (3.l0) acts on $\mathcal{H}'(k) \otimes f_k$ we get the lhs of the chain of equations (1.7) up to the step $N - 1$. It is easy to see that for the last case equation (1.7) reads

$$\hat{A}(N-1)\hat{A}^\dagger(N-1) = \lambda_1(N-1)\hat{\mathbb{I}}_{\mathcal{H}'(N)}.$$

Hence we have

$$[\mathcal{J}_+, \mathcal{J}_-] = \beta^2\lambda_1(\mathcal{J}_0 - \alpha).$$

If we put

$$\alpha = \frac{2\lambda_1(0) - \lambda_1(1)}{\lambda_1(1) - \lambda_1(0)} \qquad \beta = \sqrt{\frac{2}{|\lambda_1(0) - \lambda_1(1)|}} \tag{3.11}$$

we get

$$[\mathcal{J}_+, \mathcal{J}_-] = -2\mathcal{J}_0. \tag{3.12}$$

By (3.9) and (3.12), we conclude that $\mathcal{J}_0$ and $\mathcal{J}_\pm$ realize the representation of $su(1,1)$. Irreducible representations are realized on the subspaces $\mathcal{H}^{(j)} \subset \mathcal{H}$ ($j = N, N-1, \ldots$):

$$\mathcal{H}^{(j)} = \mathrm{span}\{\psi_{j-k}(k) \otimes f_k;\ k = j, j-1, \ldots\}. \tag{3.13}$$

Let $\Psi_{J,M} \equiv \psi_{j-k}(k) \otimes f_k$, where $J = j + \alpha$, $M = k + \alpha$. Then it is easy to prove that

$$\begin{aligned}
\mathcal{J}_+\Psi_{J,M} &= \sqrt{(M-J)(J+M+1)}\Psi_{J,M+1} \\
\mathcal{J}_-\Psi_{J,M} &= \sqrt{(M-J-1)(J+M)}\Psi_{J,M-1} \\
\mathcal{J}_0\Psi_{J,M} &= M\Psi_{J,M}
\end{aligned}$$

i.e. the irrep $D^-(J)$ of $su(1,1)$ is realized on the space $\mathrm{span}\{\Psi_{J,M};\ M = J, J-1, \ldots\}$.

4. The case $\lambda_1(0) < \lambda_1(1)$

Now the set $\{k\colon \lambda_1(k) > 0\}$ is not bounded above, but is bounded below. Sufficiently large n make $W_+(n)$ from (2.3) negative. Hence $\mathcal{H}'(0)$ is finite-dimensional as before and the expression (3.3) remains valid. By (3.2), we get

$$W_+(1) > \frac{N-1}{N}W_+(0). \tag{4.1}$$

The quantity n_0 in (3.4) is negative now. Hence an integer $N_- \leqslant 0$ can be found such that $W_+(N_-) > 0$, $W_+(N_- - 1) \leqslant 0$. N_- is the number of the first consistent model which after $-N_-$ steps generate the zero-step-considered model. So, there must be $\lambda_1(N_-) > 0$, or, equivalently, $\lambda_1(0) - n_0(\lambda_1(0) - \lambda_1(1)) \geqslant 0$. This inequality along with (3.2) gives

$$\lambda_1(0) \geqslant \frac{2}{N}W_+(0). \tag{4.2}$$

We also have the inequality (3.6). Let us suppose that it is stronger than (4.2). Then $NW_+(1) \leqslant (N-2)W_+(0)$, which is in conflict with (4.1). So, (4.2) is the only condition on $\lambda_1(0)$.

Now we shall be concerned with the algebra which will replace the chain of equations (1.7). The space $\mathcal{H}''$ is finite-dimensional now: $\mathcal{H}'' = \mathrm{span}\{f_k;\ N_- \leqslant k \leqslant N\}$. We introduce again the raising, $\mathbf{b}^\dagger$, and lowering, $\mathbf{b}$, operators on $\mathcal{H}''$ and the operators $\mathcal{J}_0$ and $\mathcal{J}_\pm$ in accordance with (3.8). Instead of (3.10) we have

$$\begin{aligned}
\beta^{-2}[\mathcal{J}_+, \mathcal{J}_-] = {} & \hat{A}(\mathcal{J}_0 - \alpha - 1)\hat{A}^\dagger(\mathcal{J}_0 - \alpha - 1) - \hat{A}^\dagger(\mathcal{J}_0 - \alpha)\hat{A}(\mathcal{J}_0 - \alpha) \\
& + \hat{A}^\dagger(\mathcal{J}_0 - \alpha)\hat{A}(\mathcal{J}_0 - \alpha)\mathcal{P}_N - \hat{A}(\mathcal{J}_0 - \alpha - 1)\hat{A}^\dagger(\mathcal{J}_0 - \alpha - 1)\mathcal{P}_{N_-}.
\end{aligned} \tag{4.3}$$

One can make sure that the appearance of the last term in the rhs of (4.3) is in accordance with the form of the chain (1.7), where $N_- - 1 \leqslant k \leqslant N - 1$, and the line corresponding to $k = N_- - 1$ is tautological: $-\hat{A}^\dagger(N_-)\hat{A}(N_-) = -\hat{A}^\dagger(N_-)\hat{A}(N_-)$. With α and β from (3.11) we get

$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_0 - \mathcal{I}(N_-) \tag{4.4}$$

where $\mathcal{I}(N_-)$ is the element of the set $\{\mathcal{I}(k)\}_{k=N_-}^N$,

$$\mathcal{I}(k) = \mathcal{P}_k[\mathcal{J}_-\mathcal{J}_+ + 2(k - N_-)\mathcal{J}_0 + \beta^2\lambda_1(N_- - 1)\mathbb{I}_\mathcal{H} - (k - N_-)(k - N_- - 1)\mathbb{I}_\mathcal{H}],$$

which obeys the relations

$$\begin{aligned} &\mathcal{J}_+\mathcal{I}(k) = \mathcal{I}(k+1)\mathcal{J}_+ \qquad \mathcal{J}_+\mathcal{I}(N) = 0 \\ &\mathcal{J}_-\mathcal{I}(k) = \mathcal{I}(k-1)\mathcal{J}_- \qquad \mathcal{J}_-\mathcal{I}(N_-) = 0. \end{aligned} \tag{4.5}$$

By analogy with the previous section we introduce the subspaces $\mathcal{H}^{(j)} \subset \mathcal{H}$, where the set of operators $\{\mathcal{J}_0, \mathcal{J}_\pm, \mathcal{I}(k)\}$ acts irreducibly:

$$\mathcal{H}^{(j)} = \text{span}\{\psi_{j-m}(N_- + m) \otimes f_{N_-+m};\ m = 0, 1, \ldots j\}. \tag{4.6}$$

Here $j = -N_-, 1 - N_-, \ldots, N - N_-$. Taking the notations $\Psi_{J,M} = \psi_{j-m}(N_- + m) \otimes f_{N_-+m}$, where $J = j + \gamma$, $M = m + \gamma$, $\gamma = \lambda_1(N_- - 1)/(\lambda_1(1) - \lambda_1(0))$, we get

$$\begin{aligned} \mathcal{J}_+\Psi_{J,M} &= \sqrt{(J - M)(J + M + 1)}\Psi_{J,M+1} \\ \mathcal{J}_-\Psi_{J,M} &= \sqrt{(J + M)(J - M + 1)}\Psi_{J,M-1} \qquad \mathcal{J}_-\Psi_{J,\gamma} = 0 \\ \mathcal{J}_0\Psi_{J,M} &= M\Psi_{J,M}. \end{aligned}$$

These relations are akin to the action of $su(2)$-generators. Note that the additional term in the rhs of (4.4) manifests itself in the values taken by J and M.

5. The case $\lambda_1(0) = \lambda_1(1)$

This situation is simple, but demands special consideration. We now have

$$W_+(n) = nW_+(1) + (1 - n)W_+(0) \qquad W_-(n) = nW_-(1) \qquad \lambda_1(k) = \lambda_1(0).$$

If $W_+(0) > W_+(1)$, the space $\mathcal{H}'(0)$ is finite-dimensional as before. There is a condition on $W_+(1)$ in this case,

$$W_+(1) = \frac{N - 1}{N}W_+(0)$$

and the positivity of $W_-(1)$ gives

$$\lambda_1(0) > \frac{W_+(0)}{N}.$$

The space $\mathcal{H}''$ and operators $\mathcal{J}_\pm$ should be taken from section 3. Taking $\beta = \lambda_1^{-1/2}(0)$, we get

$$[\mathcal{J}_+, \mathcal{J}_-] = 1.$$

This is the Heisenberg–Weyl algebra. The subspaces $\mathcal{H}^{(j)} \subset \mathcal{H}$, where $\mathcal{J}_\pm$ act irreducibly, are given by (3.13). Let us introduce the notations $\Psi_m^{(j)} \equiv \psi_{-m}(j + m) \otimes f_{j+m}$, where $m = 0, -1, -2, \ldots$. We have

$$\begin{aligned} \mathcal{J}_+\Psi_m^{(j)} &= \sqrt{-m}\Psi_{m+1}^{(j)} \\ \mathcal{J}_-\Psi_m^{(j)} &= \sqrt{-m + 1}\Psi_{m-1}^{(j)}. \end{aligned} \tag{5.1}$$

Another situation takes place when $W_+(0) < W_+(1)$. The space $\mathcal{H}'(0)$ becomes infinite-dimensional. There are no conditions on $W_+(1)$ and $\lambda_1(0)$ except the obvious positivity. The space $\mathcal{H}''$ is infinite-dimensional too (k is not bounded above). It is easy to prove that $\mathcal{J}_\pm$ when $\beta = \lambda_1^{-1/2}(0)$ have the commutator

$$[\mathcal{J}_+, \mathcal{J}_-] = 1 - \mathcal{I}(N_-) \tag{5.2}$$

where

$$\mathcal{I}(k) = \mathcal{P}_k[\mathcal{J}_-\mathcal{J}_+ + (k + 1 - N_-)\mathbb{I}_\mathcal{H}]$$

and N_- has the same sense as in the previous section. The operator $\mathcal{I}(k)$ obeys the relations (4.5)(without the number N). $\{\mathcal{J}_\pm, \mathcal{I}(k)\}$ act irreducibly on the spaces $\mathcal{H}^{(j)} \subset \mathcal{H}$ from (4.6). But now the set j is infinite: $j = -N_-, 1 - N_-, \ldots$. On the set of vectors $\Psi_m^{(j)} \equiv \psi_{-m}(N_- + j + m) \otimes f_{N_-+j+m}$, where $m = 0, -1, -2, \ldots, j$, the operators $\mathcal{J}_\pm$ act in accordance with (5.1) except for the case $m = -j$:

$$\mathcal{J}_-\Psi_{-j}^{(j)} = 0.$$

This condition (due to the additional term in the rhs of (5.2)) makes the spaces $\mathcal{H}^{(j)}$ finite-dimensional.

6. Conclusion

We have given a complete description of a special set of stochastic models (birth–death master equations). These models demonstrate hidden supersymmetry, which makes them easily solvable. The set is specified by the ansatz (2.1) and can be decomposed to four subsets depending on the ranges of parameters 1: $\lambda_1(0) > \lambda_1(1)$; 2: $\lambda_1(0) < \lambda_1(1)$; 3: $\lambda_1(0) = \lambda_1(1)$, $W_+(0) > W_+(1)$ and 4: $\lambda_1(0) = \lambda_1(1)$, $W_+(0) < W_+(1)$. It turns out that the supersymmetry chains (1.7) of these four subsets can naturally be associated with four algebras. The model of cross-inversion of enantiomers from [2] may belong to the first or second subset (depending on the relations between constant rates). This model has an additional explicit symmetry $W_+(n) = W_-(N - n)$. Hence the subsets 1 and 2 are wider and cannot be reduced to the cross-inversion model only. Note that the well-known model which describes the relaxation of quantum oscillator (see e.g. [1]) belongs to the fourth subset.

It seems that the ansatz (2.1) is the simplest. There are more complicated ones which will be described elsewhere.

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References

[1] Gardiner C W 1985 *Handbook of Stochastic Methods* (Berlin: Springer–Verlag)
[2] Il'ichov L V 1998 *Physica* A **248** 419
[3] Gendenshtein L 1983 *Pis. Zh. Teor. Fiz.* **38** 299 [*JETP Lett* **38** 356]
[4] van Kampen N G 1977 *J. Stat. Phys.* **17** 71
[5] Bender C M, Cooper F and Freedman B 1983 *Nucl. Phys.* B **219** 61
[6] Bernstein M and Brown L S 1984 *Phys. Rev. Lett.* **52** 1 933
[7] Kumar P, Ruiz-Altaba M and Thomas B S 1986 *Phys. Rev. Lett.* **57** 2 749

[8] Jauslin H R 1988 *J. Phys. A: Math. Gen.* **21** 2337
[9] Marchesoni F, Presilla C and Sodano P 1989 *Phys. Lett.* A **134** 465
[10] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (London: Gordon and Breach)
[11] Gangopadhyaya A, Mallow J V and Suhatme U P 1998 Shape invariance and its connection to potential algebra *Preprint* arXiv quant-ph/9805042